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Finite amplitude instability of second-order fluids in plane Poiseuille flow

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The hydrodynamic stability of plane Poiseuille flow of second-order fluids to finite amplitude disturbances is examined using the method of Stuart and Watson as extended by Reynolds & Potter. For slightly non-Newtonian fluids subcritical instabilities are predicted. No supercritical equilibrium states are expected if the entire spectrum of disturbance wavelengths is present. Possible implications with respect to the Toms phenomenon are discussed.

1. Introduction

The problem of hydrodynamic stability and the description of the perturbed flow is an area of importance to both theoretical fluid mechanics and practical engineering. The increasing use in plant situations of non-Newtonian fluids has added a new dimension to the problem. A well-known, but as yet not sufficiently explained, example in this area is the reduction of frictional pressure drop in pipeline flow brought about by the addition of very low concentrations of certain high molecular weight polymers to the flowing stream (the Toms phenomenon, see Toms (1948)). One postulated explanation of the drag reduction was that the transition from laminar flow was delayed by fluid viscoelastic properties. Though this transitional delay is possible in some flows (see Denn & Roisman 1969; Denn & Ginn 1969; McIntire & Schowalter 1970) other analyses have shown the opposite effect (see Chun & Schwarz 1968; Datta 1964; Walters & Thomas 1964). Experimental data (Metzner & Park 1964; Paterson & Abernathy 1970; Savins 1969; Seyer 1970; Virk & Merrill 1969; White 1970) seem to indicate that while there may be a delay in the transition region, most of the drag reduction occurs at fairly large Reynolds numbers where disturbances are no longer infinitesimal. The object of the paper is to study the nonlinear stability of plane Poiseuille flow of a slightly viscoelastic fluid. Plane Poiseuille flow is studied rather than Poiseuille flow because the method used involves the expansion of the nonlinear problem in terms of the eigenfunctions of the linearized stability problem (Eckhaus 1965). For Poiseuille flow, like plane Couette flow, no neutrally stable eigenfunctions for the linearized stability have yet been found, even for the Newtonian problem.

The technique used here to study the nonlinear stability problem was first developed in the work of Stuart (1958, 1960) and Watson (1960). Later, Reynolds & Potter (1967) proposed an extension and modification of the method of Stuart and Watson. They also carried out the numerical calculations for plane Poiseuille flow and for a combination of plane Poiseuille and plane Couette flow for a Newtonian fluid. The nonlinear analysis ultimately involves an equation for the amplitude A of velocity disturbance of the form

$$dA/dt = a^{(0)}A + a^{(2)}A^3 + \dots . \tag{1.1}$$

The first constant coefficient $a^{(0)}$ is yielded by linearized stability analysis as an eigenvalue of the Orr–Sommerfeld problem. The flow is stable to infinitesimal disturbances if $a^{(0)} < 0$ and is unstable to infinitesimal disturbances if $a^{(0)} > 0$.

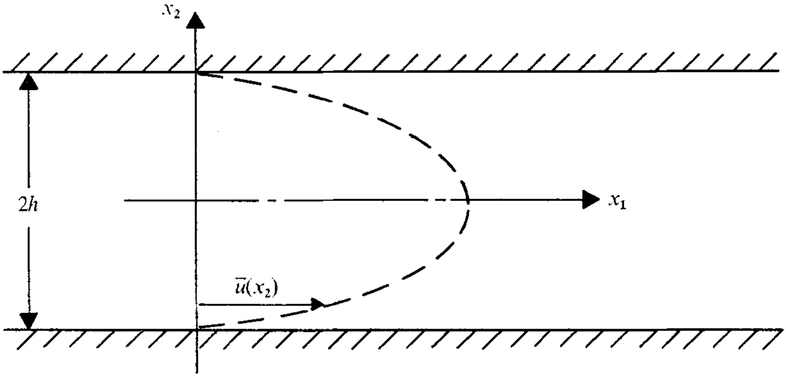

FIGURE 1. The co-ordinate system.

However, a finite amplitude disturbance might produce subcritical instability or supercritical equilibrium if the contribution of nonlinear terms outweighs the $a^{(0)}$ term. Especially in the case $a^{(0)} = 0$, corresponding to the neutral stability curve of linearized theory, the sign of $a^{(2)}$ determines whether the disturbances grow or decay.

The linearized stability analysis of plane Poiseuille flow of slightly viscoelastic fluids has been carried out by Chun & Schwarz (1968) and Bernstein & Tlapa (1970). The constitutive equation used in the work of Chun & Schwarz is the Coleman–Noll model of a second-order fluid, which is given by

$$\mathbf{S} + p\mathbf{I} = \alpha_0 \mathbf{A}_1 + \alpha_1 (\mathbf{A}_1)^2 + \alpha_2 \mathbf{A}_2, \tag{1.2a}$$

where $\alpha_0 \geqslant 0$, $\alpha_1 \geqslant 0$ and $\alpha_2 \leqslant 0$ are material constants, $\mathbf{S}$ is the stress tensor, p is the arbitrary isotropic pressure, $\mathbf{I}$ is the unit tensor and the $\mathbf{A}_i$ are the Rivlin–Ericksen tensors, defined by

$$\mathbf{A}_1 = (\nabla \mathbf{v})^T + \nabla \mathbf{v}, \tag{1.2b}$$

$$\mathbf{A}_2 = \mathscr{D}\mathbf{A}_1/\mathscr{D}t + (\mathbf{A}_1)^2, \tag{1.2c}$$

where $\mathbf{v}$ is the velocity vector and $\mathscr{D}/\mathscr{D}t$ is the Jaumann derivative. This constitutive equation is also used in the present paper. The co-ordinate system is shown in figure 1.

2. Formulation

The two-dimensional equations of motion and of continuity in Cartesian co-ordinates are given by

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=\frac{\partial s_{ji}}{\partial x_j}\quad (i=1,2), \tag{2.1}$$

$$\partial v_i/\partial x_i=0. \tag{2.2}$$

The subscript summation convention is invoked throughout the paper, except where noted. From (1.2) one obtains

$$\begin{aligned}\partial s_{ji}/\partial x_j &= \alpha_0 v_{i,jj}+\alpha_1[(v_{m,i}+v_{i,m})\,v_{m,jj}+(v_{m,ij}+v_{i,mj})\,(v_{j,m}+v_{m,j})]\\ &\quad+\alpha_2[(\partial/\partial t)\,(v_{i,jj})+2v_{m,j}(v_{j,im}+v_{i,jm})+v_m v_{i,mjj}\\ &\quad+2v_{i,mj}v_{j,m}+v_{i,m}v_{m,jj}]-\partial p/\partial x_i.\end{aligned} \tag{2.3}$$

All the variables of the problem are made dimensionless by using h, the half-width of the channel, and u_m, three halves of the bulk average velocity, viz.

$$\left.\begin{aligned}x_2=\tilde{x}_2/h,\quad x_1=\tilde{x}_1/h,\quad t=\tilde{t}u_m/h,\\ p=\tilde{p}/\rho u_m^2,\quad \mathbf{v}=\tilde{\mathbf{v}}/u_m.\end{aligned}\right\} \tag{2.4}$$

Then the characteristic Reynolds number and non-Newtonian parameters β_1 and β_2 will be defined by

$$R=u_m h\rho/\alpha_0,\quad \beta_1=\alpha_1 u_m/\alpha_0 h,\quad \beta_2=\alpha_2 u_m/\alpha_0 h. \tag{2.5}$$

Introducing the new variables

$$\theta=\alpha x_1+\omega t,\quad \omega=\omega(A),\quad A=A(t),\quad y=x_2, \tag{2.6}$$

where α is the wavenumber and ω is the frequency of the basic wave, the equation of continuity becomes

$$\alpha\frac{\partial v_1}{\partial\theta}+\frac{\partial v_2}{\partial y}=0. \tag{2.7}$$

This suggests the introduction of the stream function ψ defined by

$$\partial\psi/\partial y=\alpha v_1,\quad \partial\psi/\partial\theta=-v_2. \tag{2.8}$$

From (2.1), (2.3) and (2.8) the vorticity equation is obtained by cross-differentiation and elimination of the pressure terms. The equation is

$$\begin{aligned}&\frac{dA}{dt}\frac{\partial\zeta}{\partial A}+\left[\omega+\frac{d\omega}{dA}\left(t\frac{dA}{dt}\right)+\frac{\partial\psi}{\partial y}\right]\frac{\partial\zeta}{\partial\theta}-\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial y}-\frac{1}{R}\left(\frac{\partial^2\xi}{\partial y^2}+\alpha^2\frac{\partial^2\xi}{\partial\theta^2}\right)\\ &\quad-\frac{4\beta_2}{R}\left[\left(\frac{\partial^2\psi}{\partial y^2}-\alpha^2\frac{\partial^2\psi}{\partial\theta^2}\right)\frac{\partial^2\xi}{\partial y\,\partial\theta}+\frac{\partial^2\psi}{\partial y\,\partial\theta}\left(\alpha^4\frac{\partial^4\psi}{\partial\theta^4}-\frac{\partial^4\psi}{\partial y^4}\right)\right]=0,\end{aligned} \tag{2.9a}$$

where
$$\xi=\alpha^2\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^2\psi}{\partial y^2}, \tag{2.9b}$$

$$\zeta=\xi-\frac{\beta_2}{R}\left(\alpha^2\frac{\partial^2\xi}{\partial\theta^2}+\frac{\partial^2\xi}{\partial y^2}\right). \tag{2.9c}$$

The boundary conditions on ψ are

$$\partial\psi/\partial\theta = \partial\psi/\partial y = 0 \quad \text{at} \quad y = \pm 1. \tag{2.9d}$$

It is interesting to note that the first non-Newtonian parameter β_1 does not appear in the vorticity equation for the two-dimensional problem. The solution which is periodic in θ is sought, that is, such that ψ may be expanded in a Fourier series of θ:

$$\psi(A, y, \theta) = \Psi^{(k)} e^{ik\theta} + \tilde{\Psi}^{(k)} e^{-ik\theta}. \tag{2.10}$$

Here the summation is over all non-negative integers and $\tilde{\Psi}^{(k)}$ is the complex conjugate of $\Psi^{(k)}$. By substituting (2.10) into (2.9) and equating the coefficients of the same exponential, an infinite set of coupled nonlinear partial differential equations is obtained. The coefficient of $e^{ik\theta}$ yields

$$\begin{aligned}
&\frac{dA}{dt}\frac{\partial Z^{(k)}}{\partial A} + \left[\omega + \frac{d\omega}{dA}\left(t\frac{dA}{dt}\right)\right] ikZ^{(k)} + \frac{1}{1+\delta_{k0}}\left\{\frac{\partial\Psi^{(k-j)}}{\partial y}(ijZ^{(j)})\right. \\
&\quad - \frac{\partial\Psi^{(k+j)}}{\partial y}(ij\tilde{Z}^{(j)}) + \frac{\partial\tilde{\Psi}^{(j)}}{\partial y}[i(k+j)\,Z^{(k+j)}] - i(k-j)\,\Psi^{(k-j)}Z^{(j)} \\
&\quad \left. - i(k+j)\,\Psi^{(k+j)}\tilde{Z}^{(j)} + ij\tilde{\Psi}^{(j)}Z^{(k+j)}\right\} - \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} - \alpha^2k^2\right)\Xi^{(k)} \\
&\quad - \frac{4\beta_2}{R(1+\delta_{k0})}\left\{\left(\frac{\partial^2\Psi^{(k-j)}}{\partial y^2} + \alpha^2(k-j)^2\,\Psi^{(k-j)}\right) ij\frac{\partial\Xi^{(j)}}{\partial y}\right. \\
&\quad - \left(\frac{\partial^2\Psi^{(k+j)}}{\partial y^2} + \alpha^2(k+j)^2\,\Psi^{(k+j)}\right) ij\frac{\partial\tilde{\Xi}^{(j)}}{\partial y} + \left(\frac{\partial^2\tilde{\Psi}^{(j)}}{\partial y^2} + \alpha^2j^2\tilde{\Psi}^{(j)}\right) i(k+j)\frac{\partial\Xi^{(k+j)}}{\partial y} \\
&\quad + i(k-j)\frac{\partial\Psi^{(k-j)}}{\partial y}\left(\alpha^4j^4\Psi^{(j)} - \frac{\partial^4\Psi^{(j)}}{\partial y^4}\right) + i(k+j)\frac{\partial\Psi^{(k+j)}}{\partial y}\left(a^4j^4\tilde{\Psi}^{(j)} - \frac{\partial^4\tilde{\Psi}^{(j)}}{\partial y^4}\right) \\
&\quad \left. - ij\frac{\partial\tilde{\Psi}^{(j)}}{\partial y}\left(\alpha^4(k+j)^4\,\Psi^{(k+j)} - \frac{\partial^4\Psi^{(k+j)}}{\partial y^4}\right)\right\} = 0,
\end{aligned} \tag{2.11a}$$

where

$$\Xi^{(k)} = \left(\frac{\partial^2}{\partial y^2} - \alpha^2k^2\right)\Psi^{(k)}, \tag{2.11b}$$

$$Z^{(k)} = \Xi^{(k)} - \frac{\beta_2}{R}\left(\frac{\partial^2}{\partial y^2} - k^2\alpha^2\right)\Xi^{(k)}, \tag{2.11c}$$

$$\delta_{k0} = \begin{cases} 1 & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases} \tag{2.11d}$$

The nonlinearity and coupling of the equations would make their exact solution difficult. Therefore an approximate solution as a power series in the amplitude A is sought; that is

$$\Psi^{(k)}(A, y) = A^n\Phi^{(k,n)}(y), \tag{2.12}$$

where the superscript summation convention represents a sum over all $n \geqslant k$. Of course this approximation will be valid only if $|A|$ is small. The terms dA/dt and $\omega + (d\omega/dA)\,(t\,dA/dt)$ are also approximated by a power series, i.e.

$$A^{-1}\,dA/dt = a^{(0)} + Aa^{(1)} + A^2a^{(2)} + \ldots = A^na^{(n)}, \tag{2.13}$$

$$\omega + (d\omega/dA)\,(t\,dA/dt) = b^{(0)} + Ab^{(1)} + \ldots = A^nb^{(n)}. \tag{2.14}$$

Substituting (2.12)–(2.14) into (2.11) and equating the terms of the same order gives an infinite set of equations for $\phi^{(k,n)}$. These equations are then rearranged by collecting the terms involving $\phi^{(k,n)}$, and replacing the terms involving $\phi^{(0,0)}$ by the basic laminar velocity $\bar{u}$ according to the relation

$$\bar{u} = 2\alpha^{-1}D\phi^{(0,0)} = 2\alpha^{-1}D\tilde{\phi}^{(0,0)}, \tag{2.15}$$

where $D = d/dy$. The final set of equations may be written as

$$L_{kn}\phi^{(k,n)} = i\alpha c^{(n-1)}G\delta_{k1} + H_{kn} \quad \text{(no sum)}. \tag{2.16a}$$

Here δ_{kj} is the Kronecker delta, the operator L_{kn} is defined by

$$\begin{aligned} L_{kn} = {} & ik[-i(n/k)\,a^{(0)} + b^{(0)} + \alpha\bar{u}]\,[D^2 - \alpha^2k^2 - (\beta_2/R)\,(D^2 - k^2\alpha^2)^2] \\ & - \alpha(D^2\bar{u} - (\beta_2/R)\,D^4\bar{u}) - (1/R)\,(D^2 - k^2\alpha^2)^2 \end{aligned} \tag{2.16b}$$

and the other terms are defined by

$$i\alpha c^{(n)} = -(a^{(n)} + ib^{(n)}), \tag{2.16c}$$

$$G = [(D^2 - \alpha^2) - (\beta_2/R)\,(D^2 - \alpha^2)^2]\,\phi^{(1,1)}, \tag{2.16d}$$

$$\begin{aligned} H_{kn} = {} & -(ma^{(n-m)} + ikb^{(n-m)})\,[D^2 - k^2\alpha^2 - (\beta_2/R) \\ & \times (D^2 - k^2\alpha^2)^2]\,\phi^{(k,m)} + F_{kn}/1 + \delta_{k0}, \end{aligned} \tag{2.16e}$$

$$\begin{aligned} F_{kn} = {} & -(D\phi^{(k-j,n-m)})\,(ij[D^2 - j^2\alpha^2 - (\beta_2/R)\,(D^2 - j^2\alpha^2)^2]\,\phi^{(j,m)}) \\ & + (D\phi^{(k+j,n-m)})\,(ij[D^2 - j^2\alpha^2 - (\beta_2/R)\,(D^2 - j^2\alpha^2)^2]\,\tilde{\phi}^{(j,m)}) \\ & - (D\tilde{\phi}^{(j,n-m)})\,(i(k+j)\,[D^2 - (k+j)^2\alpha^2 - (\beta_2/R)\,(D^2 - (k+j)^2\alpha^2)^2]\,\phi^{(k+j,m)}) \\ & + (i(k-j)\,\phi^{(k-j,n-m)})\,([D^3 - j^2\alpha^2D - (\beta_2/R)\,D(D^2 - j^2\alpha^2)^2]\,\phi^{(j,m)}) \\ & + (i(k+j)\,\phi^{(k+j,n-m)})\,([D^3 - j^2\alpha^2D - (\beta_2/R)\,D(D^2 - j^2\alpha^2)^2]\,\tilde{\phi}^{(j,m)}) \\ & - (ij\tilde{\phi}^{(j,n-m)})\,([D^3 - (k+j)^2\alpha^2D^2 - (\beta_2/R)\,D(D^2 - (k+j)^2\alpha^2)^2]\,\phi^{(k+j,m)}) \\ & + (4\beta_2/R)\,\{([D^2 + (k-j)^2\alpha^2]\,\phi^{(k-j,n-m)})\,(ij[D^3 - j^2\alpha^2D]\,\phi^{(j,m)}) \\ & - ([D^2 + (k+j)^2\alpha^2]\,\phi^{(k+j,n-m)})\,(ij[D^3 - j^2\alpha^2D]\,\tilde{\phi}^{(j,m)}) \\ & + ([D^2 + j^2\alpha^2]\,\tilde{\phi}^{(j,n-m)})\,(i(k+j)\,[D^3 - (k+j)^2\alpha^2D]\,\phi^{(k+j,m)}) \\ & - (i(k-j)\,D\phi^{(k-j,n-m)})\,([D^4 - j^4\alpha^4]\,\phi^{(j,m)}) \\ & - (i(k+j)\,D\phi^{(k+j,n-m)})\,([D^4 - j^4\alpha^2]\,\tilde{\phi}^{(j,m)}) \\ & + (ijD\tilde{\phi}^{(j,n-m)})\,([D^4 - (k+j)^4\alpha^4]\,\phi^{(k+j,m)})\}. \end{aligned} \tag{2.16f}$$

Here, the superscript summation convention is used and the rule is the same as the one discussed in the work of Reynolds & Potter (1967). The boundary conditions become

$$\phi^{(k,n)} = D\phi^{(k,n)} = 0 \quad \text{at} \quad y = \pm 1. \tag{2.16g}$$

For the case $k = n = 1$, the problem is equivalent to linearized stability problem, i.e.

$$\begin{aligned} \{ & (a^{(0)} + ib^{(0)} + i\alpha\bar{u})\,[D^2 - \alpha^2 - (\beta_2/R)\,(D^2 - \alpha^2)^2] - i\alpha[D^2\bar{u} - (\beta_2/R)\,D^4\bar{u}] \\ & - 1/R\,(D^2 - \alpha^2)^2 - (4i\alpha\beta_2/R)\,[D\bar{u}(D^3 - \alpha^2D) - D^3\bar{u}D]\}\,\phi^{(1,1)} = 0, \end{aligned} \tag{2.17a}$$

$$\phi^{(1,1)} = D\phi^{(1,1)} = 0 \quad \text{at} \quad y = \pm 1. \tag{2.17b}$$

The coefficients $a^{(0)}$ and $b^{(0)}$ are related to the conventional eigenvalues of the linearized problem by

$$b^{(0)} = -\alpha c_r, \quad a^{(0)} = \alpha c_i. \tag{2.18}$$

The mean velocity profile of the basic flow is the same as for a Newtonian fluid, i.e.

$$\bar{u} = (1-y^2). \tag{2.19}$$

It is then easily seen that the operator in (2.17 a) is even, thus its eigensolutions can be separated into even and odd modes. As in the Newtonian fluid problem, the most dominant eigenvalue of (2.17) is with even mode. Therefore only the symmetric disturbance will be considered here. The boundary conditions for even $\phi^{(1,1)}$ are replaced by

$$D\phi^{(1,1)} = D^3\phi^{(1,1)} = 0 \quad \text{at} \quad y = 0. \tag{2.20 a}$$

Since any multiple of an eigenfunction is also a solution of (2.17), $\phi^{(1,1)}$ can be normalized according to

$$\phi^{(1,1)} = 1 \quad \text{at} \quad y = 0. \tag{2.20 b}$$

If $\phi^{(1,1)}$ is known, the higher order problems can be solved sequentially. By considering the interaction of nonlinear terms it can be shown that

$$\phi^{(k,n)} = 0 \quad \text{if} \quad k+n = \text{odd}, \tag{2.21 a}$$

and that

$$\phi^{(k,n)} \quad \text{is} \quad \begin{Bmatrix} \text{odd for even } n, \\ \text{even for odd } n. \end{Bmatrix} \tag{2.21 b}$$

This provides the central boundary conditions for the higher order problems.

3. Results

3.1. *Linear problems*

Chun & Schwarz (1968) have carried out numerical calculations for the linearized problem and have obtained neutral stability curves. Unfortunately, the last two terms in the left-hand side of (2.17 a) were omitted in their disturbance equation. A numerical scheme used by Landahl & Kaplan (1965) is employed here to integrate equation (2.17). This numerical procedure was tested by calculating results for the special case $\beta_2 = 0$ (Newtonian fluid) and comparing these with the numerical results obtained by Thomas (1953). While Thomas found

$$c = 0{\cdot}2375259 + 0{\cdot}0037404i$$

for the case $R = 10\,000$, $\alpha = 1{\cdot}0$, the present calculation gave

$$c = 0{\cdot}2375267 + 0{\cdot}0037399i.$$

The numerical values of the eigenfunction were essentially identical.

The curves of neutral stability ($a^{(0)} = 0$) are shown in figure 2 for different values of β_2. They show quantitatively different but qualitatively similar results to the ones obtained by Chun & Schwarz. The presence of the non-Newtonian parameter β_2 moves the neutral stability curve toward the left and therefore decreases the critical Reynolds number in linear stability analysis.

The eigenvalue problem (2.17) can also be solved by approximation methods such as the Galerkin method (see Finlayson 1968) and the variational method (Lee & Reynolds 1967). Since the adjoint eigenvalue problem to (2.17) has the same boundary conditions, these two methods are essentially equivalent and would yield the identical results. Although these approximation methods yield less accurate eigenvalues and eigenfunctions they give the spectrum of the eigenvalues, which is important in finding the most dominant eigenvalue. The Galerkin method is used here to examine the eigenvalue spectrum of the problem. It is observed that for non-Newtonian fluids, if R is fixed and α is increased, then there is an eigenvalue which changes from extremely stable to extremely unstable.

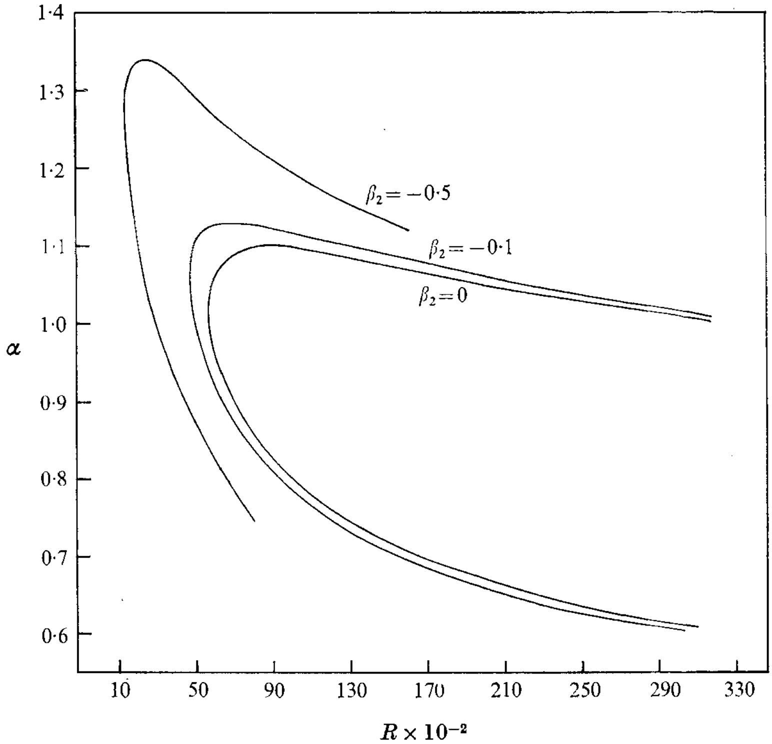

FIGURE 2. Neutral stability curves for the linearized case.

This could be explained as follows (see Shen 1964). Multiplying the (2.17) by the conjugate eigenfunction $\tilde{\phi}^{(1,1)}$ and integrating it from $y = -1$ to $y = +1$, one obtains

$$a^{(0)} = \frac{-(1/R)(I_2^2+2\alpha_2 I_1^2+\alpha^4 I_0^2)+\alpha(Q-\tilde{Q})/2i}{(I_1^2+\alpha^2 I_0^2)+(\beta_2/R)(I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2)}, \tag{3.1a}$$

where
$$I_0 = \int_{-1}^{1} \phi^{(1,1)}\tilde{\phi}^{(1,1)}\,dy, \tag{3.1b}$$

$$I_1 = \int_{-1}^{+1} D\phi^{(1,1)}\,D\tilde{\phi}^{(1,1)}\,dy, \tag{3.1c}$$

$$I_2 = \int_{-1}^{+1} D^2\phi^{(1,1)}\,D^2\tilde{\phi}^{(1,1)}\,dy, \tag{3.1d}$$

$$Q = \int_{-1}^{+1}\left[(1-y^2)\left(D^2-\alpha^2-\frac{\beta_2}{R}(D^2-\alpha^2)^2\right)+2 + \frac{8\beta_2}{R}y(D^3-\alpha^2 D)\right]\phi^{(1,1)}\tilde{\phi}^{(1,1)}\,dy. \tag{3.1e}$$

It can be easily seen that when $\beta_2 = 0$ the denominator on the right-hand side of (3.1 *a*) will never vanish. However, if $\beta_2 < 0$, which is the case for a second-order fluid, then an eigenfunction such that the denominator vanishes may exist. This will cause the eigenvalue associated with it to behave abnormally. Eventually, it is found that for any given negative value of β_2 there always exists an abnormally behaved eigenvalue for any R and α, if enough terms of approximation are adopted. This has caused some argument about the existence of a neutral stability curve (see Craik 1968; Platten & Schechter 1970) and should be considered as a shortcoming of using the constitutive equation (1.2 *a*) for a slightly viscoelastic fluid. However, in the present paper, only the normally behaving eigenvalues (i.e. those for which the denominator does not vanish) are considered.

3.2. *Nonlinear analysis*

Once $\phi^{(1,1)}$ is known, the right-hand side of the equations for $\phi^{(0,2)}$ and $\phi^{(2,2)}$ can be calculated. These inhomogeneous linear equations are then integrated by Landahl & Kaplan's scheme to give $\phi^{(0,2)}$ and $\phi^{(2,2)}$. Before moving on to calculate $\phi^{(1,3)}$, the constant $c^{(2)}$ has to be determined. This can be done with the help of the adjoint eigensolution. Stuart (1960) and Reynolds & Potter (1967) showed that $c^{(n)}$ can be calculated according to the relation

$$i\alpha c^{(n-1)} = a^{(n-1)} + ib^{(n-1)} = \int_{-1}^{+1} H_{1n}\,\Phi\,dy \Big/ \int_{-1}^{+1} G\Phi\,dy, \tag{3.2}$$

where Φ is the solution of the adjoint eigenvalue problem to (2.16), i.e.

$$L_{11}^{*}\Phi = 0, \quad B^{*}\Phi = 0, \tag{3.3}$$

where L_{11}^{*} is the adjoint operator to L_{11}, and B^{*} are the corresponding boundary conditions. L_{11}^{*} can be easily obtained by integration by parts and is given by

$$\begin{aligned} L_{11}^{*} = {} & (a^{(0)} + ib^{(0)} + i\alpha\bar{u})\,[D^2 - \alpha^2 - (\beta_2/R)\,(D^2-\alpha^2)^2] + 2D\bar{u}D \\ & - 1/R\,(D^2-\alpha^2)^2 - (i\alpha\beta_2/R)\,(D^4\bar{u} - 4D^3\bar{u}D - 6D^2\bar{u}D^2 + 2\alpha^2 D^2\bar{u}). \end{aligned} \tag{3.4}$$

The boundary conditions B^{*} are identical with those given by (2.17 *b*). The adjoint eigenfunction Φ is also integrated by using Landahl & Kaplan's scheme. The adjoint eigenvalue should be identical with the original problem and this serves as a criterion for the accuracy of the numerical scheme. The numerical calculation using a single precision program was carried out on a Burroughs-5500. The results for a Newtonian fluid are given in table 1 and are consistent with the ones obtained by Reynolds & Potter. The results for two non-Newtonian cases ($\beta_2 = -0{\cdot}1,\ -0{\cdot}5$) are summarized in tables 2 and 3.

The results reveal that for plane Poiseuille flow the presence of the non-Newtonian parameter β_2 does not have a qualitative effect on the value of $c^{(2)}$. In the neighbourhood of the critical point, $a^{(2)}$ appears to be positive for both $\beta_2 = -0{\cdot}1$ and $\beta_2 = -0{\cdot}5$. Therefore a subcritical instability is also conjectured for the non-Newtonian fluid. While on the upper branch of the neutral stability curve $a^{(2)}$ is positive and becomes larger when the fluid becomes more non-Newtonian, $a^{(2)}$ is negative on the lower branch and becomes smaller when β_2 decreases. Hence

R	α	c_r	c_i	$a^{(2)}$	$b^{(2)}$	K_1	K_2	K_3
30 000	1·0067	0·1948	0·0000	430·0	−809·8	−18·36	−28·70	907·1
20 000	1·0471	0·2132	0·0000	314·8	−600·0	−14·84	−21·91	666·4
12 000	1·0865	0·2377	0·0000	195·3	−398·0	−10·61	−13·24	414·5
9 000†	1·097	0·2515	0·0000	136·1	−309·0	−8·36	−8·57	239·5
7 500†	1·094	0·2597	0·0000	101·2	−258·0	−6·94	−5·96	214·6
5 772·2†‡	1·02071	0·2640	0·0000	29·67	−165·9	−4·23	−2·70	66·0
7 500†	0·875	0·2345	0·0000	−4·91	−129·0	−3·20	−3·99	−2·61
9 000†	0·823	0·2203	0·0000	−10·63	−133·8	−3·06	−4·61	−13·59
12 000	0·7576	0·2007	0·0000	−15·03	−120·9	−2·972	−5·279	−21·82
20 000	0·6673	0·1714	0·0000	−18·80	−124·9	−2·952	−6·067	−28·59
30 000	0·6095	0·1517	0·0000	−20·18	−132·9	−2·998	−6·494	−30·86
5 250†	1·02071	0·2684	−0·0015	33·03	−163·5	−5·23	−2·51	73·8
6 000†	1·02071	0·2263	0·0005	29·83	−170·0	−4·08	−2·78	66·6
20 000	0·750	0·1823	0·00220	−44·35	−160·0	−2·422	−4·633	−81·66
20 000	0·800	0·1866	0·00727	−31·30	−188·5	−2·761	−4.46	−55·4
20 000	0·834	0·1926	0·00782	−15·05	−209·6	−3·036	−4·816	−22·25
20 000	0·867	0·1964	0·00786	6·47	−232·2	−3·354	−5·608	21·90
20 000	0·900	0·2000	0·00742	34·46	−258·6	−3·763	−6·907	79·58
30 000	0·7088	0·1645	0·00696	−57·67	−181·4	−2·523	−4·816	−108·0
30 000	0·8081	0·1764	0·00915	−8·59	−251·1	−3·332	−6·065	−7·784
30 000	0·9074	0·1868	0·00675	115·9	−362·1	−4·942	−12·54	249·4

† Data from Reynolds & Potter. ‡ Critical point

TABLE 1. Summary of results for Newtonian fluid ($\beta_2 = 0$)

R	α	c_r	c_i	$a^{(2)}$	$b^{(2)}$	K_1	K_2	K_3
30 000	1·0139	0·1948	0·0000	438·1	−829·4	−19·57	−31·72	927·6
20 000	1·0575	0·2134	0·0000	326·1	−618·8	−16·06	−25·07	693·3
12 000	1·1036	0·2384	0·0000	210·5	−416·5	−11·79	−16·37	449·1
9 000	1·1212	0·2530	0·0000	154·8	−327·1	−9·495	−11·64	330·8
7 500	1·1265	0·2620	0·0000	122·0	−277·7	−8·089	−8·809	260·9
6 000	1·1209	0·2723	0·0000	82·42	−222·1	−6·382	−5·622	176·8
4 886†	1·0501	0·2751	0·0000	25·88	−153·6	−4·186	−2·885	58·76
6 000	0·9186	0·2494	0·0000	−4·50	−121·9	−3·194	−3·833	−1·976
7 500	0·8505	0·2311	0·0000	−11·79	−114·9	−3·007	−4·575	−15·99
9 000	0·8057	0·2178	0·0000	−14·94	−112·6	−2·936	−5·021	−21·93
12 000	0·7461	0·1991	0·0000	−17·78	−112·4	−2·891	−5·548	−27·10
20 000	0·6607	0·1704	0·0000	−20·15	−118·7	−2·906	−6·189	−31·20
30 000	0·6050	0·1511	0·0000	−21·04	−127·7	−2·966	−6·560	−32·55
5 000	1·050	0·2738	0·00025	26.25	−155·7	−4·052	−2·958	59·51
4 500	1·050	0·2788	−0·00078	26·57	−149·6	−4·639	−2·717	60·49
20 000	0·750	0·1822	0·00611	−47·74	−157·0	−2·426	−4·774	−88·28
20 000	0·800	0·1884	0·00790	−34·43	−185·5	−2·778	−4·719	−61·80
20 000	0·834	0·1924	0·00847	−18·18	−206·3	−3·052	−5·169	−28·13
20 000	0·867	0·1962	0·00852	3·161	−228·4	−3·367	−6·063	15·75
20 000	0·900	0·1998	0·00810	30·61	−253·2	−3·765	−7·470	72·4
30 000	0·7088	0·1645	0·00742	−60·26	−178·7	−2·541	−4·945	−113·0
30 000	0·8081	0·1762	0·00965	−11·03	−248·0	−3·355	−6·440	−12·25
30 000	0·9074	0·1865	0·00730	110·7	−354·3	−4·926	−1·325	239·6

† Critical point.

TABLE 2. Summary of results for non-Newtonian fluid I ($\beta_2 = -0{\cdot}1$)

a finite amplitude equilibrium would be possible only if the disturbance could be limited to very low wavenumbers. For a fixed R (for example $R = 20\,000$), starting at the lower neutral stability point where $a^{(2)}$ is negative, as α increases $a^{(2)}$ first descends to a minimum and then increases and becomes positive when α exceeds a certain value. The variation of $a^{(2)}$ with respect to α is illustrated in figure 3. The constants K_1, K_2, and K_3, which are also given in tables 1–3,

R	α	c_r	c_i	$a^{(2)}$	$b^{(2)}$	K_1	K_2	K_3
9000	1·210	0·2556	0·0000	214·4	−398·7	−16·23	−29·36	474·4
6000	1·262	0·2796	0·0000	158·7	−295·9	−13·09	−22·90	353·4
3000	1·337	0·3242	0·0000	85·53	−173·5	−8·169	−12·85	192·1
1579†	1·280	0·3602	0·0000	11·68	−95·37	−3·501	−4·736	31·60
3000	0·9847	0·2902	0·0000	−21·40	−70·81	−2·562	−5·299	−34·94
6000	0·8255	0·2367	0·0000	−23·32	−73·18	−2·551	−5·795	−38·30
9000	0·7530	0·2104	0·0000	−23·30	−78·76	−2·599	−6·027	−37·96
1400	1·280	0·3679	−0·00108	6·14	−87·99	−3·431	−4·420	20·13
1800	1·280	0·3519	0·00093	18·83	−103·9	−3·683	−5·287	46·63

† Critical point.

TABLE 3. Summary of results for non-Newtonian fluid II ($\beta = -0{\cdot}5$)

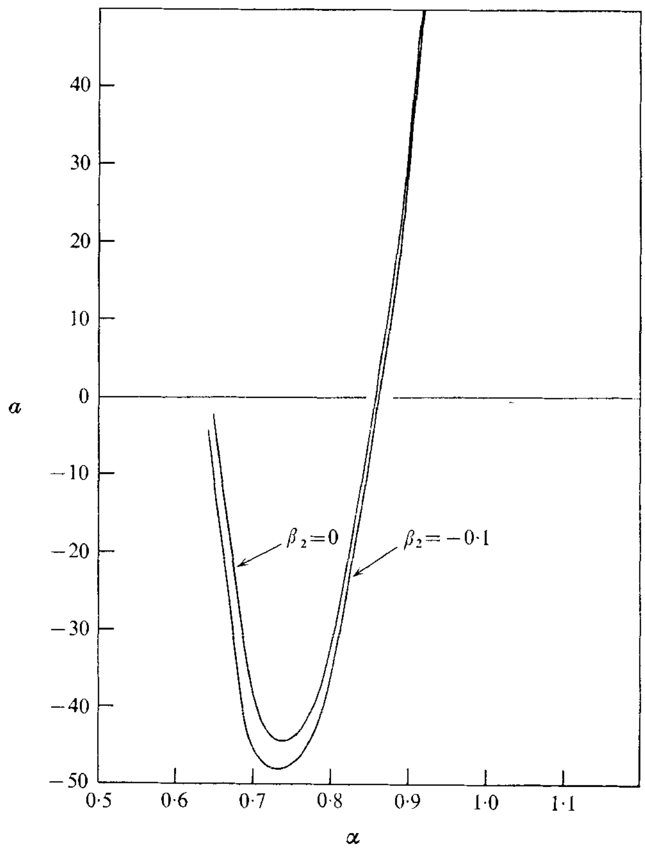

FIGURE 3. The finite amplitude correction as a function of wavenumber at a Reynolds number of 20000.

represent the contributions to $a^{(2)}$ due to the following three physical processes: (i) the distortion of the mean motion (K_1); (ii) the generation of the harmonic of the fundamental (K_2); (iii) the distortion of the y-dependence of the fundamental (K_3). These three constants are defined by equations (6.3)–(6.5) in Stuart (1960). They are related to $a^{(2)}$ by

$$2a^{(2)} = K_1 + K_2 + K_3. \tag{3.5}$$

The normalization procedures imply that the total mean flow is held constant. In this case the mean pressure gradient in θ direction does change with time. The following expansion is adopted:

$$-\partial p/\partial\theta = \pi^{(k)} e^{ik\theta} + \tilde{\pi}^{(k)} e^{-ik\theta}. \tag{3.6}$$

By substituting (2.8) and (3.6) into equations of motion it can be shown that for the mean pressure gradient ($k = 0$)

$$\begin{aligned}\pi^{(0)} = \frac{1}{\alpha^2}\Bigg\{&\frac{dA}{dt}\frac{\partial}{\partial A}\left[\frac{\partial\Psi^{(0)}}{\partial y} - \frac{\beta_2}{R}\frac{\partial^3\Psi^{(0)}}{\partial y^3}\right] - \frac{ij}{2}\left(\Psi^{(j)}\frac{\partial^2}{\partial y^2}\tilde{\Psi}^{(j)} - \tilde{\Psi}^{(j)}\frac{\partial^2}{\partial y^2}\Psi^{(j)}\right)\\
&-\frac{1}{R}\left[\frac{\partial^3\Psi^{(0)}}{\partial y^3} + \beta_2\left(ij^3\alpha^2\left(\frac{\partial^2\Psi^{(j)}}{\partial y^2}\tilde{\Psi}^{(j)} - \frac{\partial^2\tilde{\Psi}^{(j)}}{\partial y^2}\Psi^{(j)}\right)\right.\right.\\
&\left.\left.+2ij\left(\frac{\partial^3\Psi^{(j)}}{\partial y^3}\tilde{\Psi}^{(j)} - \frac{\partial^3\tilde{\Psi}^{(j)}}{\partial y^3}\Psi^{(j)}\right) - \frac{ij}{2}\left(\Psi^{(j)}\frac{\partial^4\tilde{\Psi}^{(j)}}{\partial y^4} - \tilde{\Psi}^{(j)}\frac{\partial^4\Psi^{(j)}}{\partial y^4}\right)\right)\right]\Bigg\}\end{aligned} \tag{3.7 a}$$

and

$$\partial\pi^{(0)}/\partial y = 0. \tag{3.7 b}$$

Since the functions $\Psi^{(j)}$ have been expanded in a power series in A, from (3.7), $\pi^{(0)}$ can also be expanded into a power series in A, i.e.

$$\pi^{(0)} = \pi^{(0,n)} A^n, \tag{3.8}$$

where the $\pi^{(0,n)}$ are constant and independent of y because of (3.7 b). Putting (2.13) and (3.8) into (3.7 a), one obtains

$$\begin{aligned}\pi^{(0,n)} = \frac{1}{\alpha^2}\Bigg\{&m a^{(n-m)}\left(D\phi^{(0,m)} - (\beta_2/R)\, D^3\phi^{(0,m)}\right)\\
&-\tfrac{1}{2}ij\left(\phi^{(j,n-m)} D^2\tilde{\phi}^{(j,m)} - \tilde{\phi}^{(j,n-m)} D^2\phi^{(j,m)}\right)\\
&-\frac{1}{R}\Big[D^3\phi^{(0,n)} + \beta_2\Big\{ij^3\alpha^2\left(D^2\phi^{(j,n-m)}\tilde{\phi}^{(j,m)} - D^2\tilde{\phi}^{(j,n-m)}\phi^{(j,m)}\right)\\
&+2ij\left[D^3\phi^{(j,n-m)}\tilde{\phi}^{(j,m)} - D^3\tilde{\phi}^{(j,n-m)}\phi^{(j,m)}\right.\\
&\left.-\tfrac{1}{2}ij\left(\phi^{(j,n-m)} D^4\tilde{\phi}^{(j,m)} - \tilde{\phi}^{(j,n-m)} D^4\phi^{(j,m)}\right)\right]\Big\}\Big]\Bigg\}.\end{aligned} \tag{3.9}$$

For $n = 0$, by using (2.15) and (3.9), the undisturbed mean pressure gradient becomes

$$\partial\bar{p}/\partial x_1 = -2\alpha\pi^{(0,0)} = -2/R. \tag{3.10}$$

$\pi^{(0,2)}$, which can be calculated if $\phi^{(0,2)}$ and $\phi^{(1,1)}$ are known, is the $O(A^2)$ contribution to the mean pressure gradient. The numerical results for $\pi^{(0,2)}$ are summarized in table 4. The presence of β_2 tends to increase the $O(A^2)$ contribution to the mean pressure gradient. Pipkin and Walters (see Pipkin 1964*a*, *b*; Jones & Walters 1967; Walters & Townsend 1970) have examined the effect of pressure oscillation of non-Newtonian fluids in Poiseuille flow.

R	β_2	α	c_i	$\pi^{(0,2)} \times 100$
7 500	0	1·0944	0·0000	2·042
7 500	−0·1	1·1265	0·0000	2·123
20 000	0	0·867	0·00786	3·393
20 000	−0·1	0·867	0·00852	3·430
30 000	0	0·6095	0·0000	1·582
30 000	−0·1	0·6050	0·0000	1·621
40 000	0	0·5737	0·0000	1·510
40 000	0	0·9755	0·0000	1·481
40 000	−0·1	0·9755	0·00052	2·855
40 000	−0·1	0·5401	0·0000	1·527
40 000	−0·1	0·9810	0·0000	1·512

TABLE 4. $O(A^2)$ contribution to mean pressure gradient

4. Conclusions

The behaviour of slightly viscoelastic fluids in plane Poiseuille flow with finite amplitude disturbances is not greatly different from that of Newtonian fluids. Subcritical instabilities are still predicted and, if the entire spectrum of disturbance wavelengths is present, no supercritical equilibrium flows appear possible. This would indicate that the explanation of drag reduction in viscoelastic fluids is to be found in the effect of non-Newtonian parameters on the flow far from the neutral stability curve (in the eddying turbulent field). No dramatic effects of non-Newtonian parameters on the creation of equilibrium flows for small but finite disturbances are found.

At constant mean flow rate the effect of the perturbation oscillation in plane Poiseuille flow is to increase the average pressure gradient for second-order fluids – relative to the Newtonian value.

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